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Intrinsic characterisation of orthogonal *R* separation for Laplace equations

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Abstract. This paper gives a coordinate-free characterisation of R separation for the Laplace equation on a pseudo-Riemannian manifold in terms of commuting conformal symmetry operators. The coordinates can be computed from a knowledge of the operators.

1. Introduction

Let V_n be an n-dimensional pseudo-Riemannian manifold and let $\{y^j\}$ be a local coordinate system for V_n. The *Laplace* (or wave) *equation* is

$$\Delta\psi(y)=0 \tag{1.1}$$

where ψ is a function on V_n and Δ is the Laplace–Beltrami operator (Eisenhart 1949)

$$\Delta=\frac{1}{\sqrt{g}}\sum_{i,j=1}^{n}\partial_i(\sqrt{g}\,g^{ij}\partial_j). \tag{1.2}$$

Here, $\partial_j=\partial_{y^j}$, the metric on V_n, expressed in terms of the coordinates $\{y^j\}$, is

$$\mathrm{d}s=\sum_{i,j} g_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j \qquad g=\det(g_{ij})$$

and $\sum_k g^{ik}g_{kj}=\delta^i_j$. Associated with the Laplace equation is the Hamilton–Jacobi equation (Kalnins and Miller 1982a)

$$\mathcal{H}(\partial_i W)\equiv\sum_{i,j=1}^{n} g^{ij}\partial_i W\partial_j W=0 \tag{1.3}$$

where W is a function on V_n and $\mathcal{H}$ is the *Hamiltonian* function

$$\mathcal{H}(p_i)=\sum_{i,j=1}^{n} g^{ij}p_ip_j \tag{1.4}$$

defined on the cotangent bundle $\tilde{V}_n$. Both Δ and $\mathcal{H}$ are coordinate-independent objects.

As is well known (Moon and Spencer 1952, Kalnins and Miller 1978) there is a close relationship between additively separable solutions of (1.3) and multiplicative

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R-separable solutions of the Laplace equation:

$$W(y)=\sum_{j=1}^{n} W^{(j)}(y^j,\boldsymbol{\lambda}) \qquad \psi(y)=R(y)\prod_{j=1}^{n}\psi^{(j)}(y^j,\boldsymbol{\lambda}). \tag{1.5}$$

(Here $\lambda_2,\ldots,\lambda_n$ are separation constants. If $R\equiv 1$ we have ordinary separation and if $\partial_{ij}\ln R=0$ for $i\neq j$ we have trivial R separation.) In particular, every R-separable coordinate system for (1.1) can be shown to be also separable for (1.3).

In Kalnins and Miller (1982a) the authors characterised additive separation for the Hamilton–Jacobi equation (1.3) in terms of involutive families of conformal Killing tensors for $\tilde{V}_n$. Here we shall find the analogous characterisation of R separation for the Laplace equation (1.1) in terms of families of commuting conformal symmetry operators. Briefly, we shall show that a separable system $\{y^j\}$ for (1.3) is R separable for (1.1) if and only if the system of defining conformal Killing tensors associated with $\{y^j\}$ can be extended to a family of commuting conformal symmetry operators. Thus in essence we are dealing with a problem in quantisation theory.

In Kalnins and Miller (1980, 1982b) we solved the corresponding problems for the Hamilton–Jacobi equation $\mathscr{H}=E$ and the Helmholtz (or Schrödinger) equation $\Delta\psi=E\psi$, reaching similar conclusions. However, the techniques of those papers do not extend to the present case in a straightforward manner. For the Helmholtz equation it turned out that there was a unique family of symmetry operators associated with an R-separable coordinate system $\{y^j\}$. In the present case there may be several such families and this lack of uniqueness greatly complicates the theory. Fortunately our final result, theorem 3, has a satisfying degree of simplicity and elegance.

In § 2 we give a precise constructive definition of orthogonal R separation for the Laplace equation and work out the technical conditions for the success of the construction. Then in § 3 we determine the symmetry operator significance of these technical conditions.

The results of this paper have obvious application to all fields in which explicit solutions of Laplace and wave equations on manifolds are relevant. For some of the details of the applications, see Miller (1977).

All our considerations are local rather than global although it is clear that our results can be extended to construct a global theory of variable separation. Any function occurring in this paper is assumed to be locally analytic.

2. Orthogonal R separation

Let $\{x^i\}$ be an orthogonal coordinate system on the (local) pseudo-Riemannian manifold V_n. In these coordinates the metric is

$$ds'^2=\sum_{i=1}^{n} H_i'^2(dx^i)^2 \tag{2.1}$$

and the Laplace equation becomes

$$\Delta\psi\equiv\frac{1}{h'}\sum_i \partial_i(h'H_i'^{-2}\partial_i\psi)=0 \tag{2.2}$$

where $h' = H'_1 \ldots H'_n$. We briefly review the construction to obtain R-separable solutions

$$\psi(\boldsymbol{x}) = R(\boldsymbol{x}) \prod_{j=1}^{n} \psi^{(j)}(x^j) \tag{2.3}$$

for (2.2) and find necessary and sufficient conditions for the success of the construction. Let $(S_{ji}(x^j))$ be a *Stäckel matrix*, i.e. an $n \times n$ non-singular matrix whose jth row depends only on the variable x^j, and set $S = \det(S_{ji})$. Let $\lambda_2, \ldots, \lambda_n$ be complex parameters and define differential operators K_j, L_j, $j = 1, \ldots, n$ by

$$K_j = L_j + \sum_{j=1}^{n} \lambda_i S_{ji}(x^j) \qquad L_j = \partial_{jj} + l_j \partial_j + m_j \tag{2.4}$$

where l_j, m_j are functions of x^j alone. We say that the orthogonal coordinates $\{x^i\}$ are *R separable* for the Laplace equation (2.2) provided there exist functions $g_i(\boldsymbol{x})$ and $R(\boldsymbol{x})$, $(R \neq 0)$, such that

$$R^{-1} \Delta R \equiv \sum_{j=1}^{n} g_j(\boldsymbol{x}) K_j. \tag{2.5}$$

(Here

$$R^{-1} \Delta R = \frac{1}{h'} \sum_i \partial_i (h' H_i'^{-2} \partial_i) + 2 \sum_i H_i'^{-2} (\partial_i \ln R) \partial_i + R^{-1}(\Delta R) \tag{2.6}$$

is an operator.) If these coordinates are R separable then the function ψ, (2.3), is a solution of $\Delta\psi = 0$ whenever the $\psi^{(j)}$ satisfy the (ordinary differential) *separation equations*

$$K_j \psi^{(j)} = 0 \qquad j = 1, \ldots, n. \tag{2.7}$$

A simple consequence of (2.5) and (2.6) is the necessary condition for R separation

$$H_j'^{-2} = Q(\boldsymbol{x}) S^{j1}/S \tag{2.8}$$

where S^{j1} is the $(j, 1)$ minor of (S_{ji}) and Q is a non-zero function. Hence the metric $\mathrm{d}s'^2$ is in *conformal Stäckel form* and the metric $\mathrm{d}s^2$ is in *Stäckel form*, where

$$\mathrm{d}s'^2 = Q^{-1}\, \mathrm{d}s^2 = Q^{-1} \sum_j H_j^2 (\mathrm{d}x^j)^2 \qquad H_j^{-2} = S^{j1}/S. \tag{2.9}$$

It follows directly from the work of Stäckel himself (1891) that condition (2.8) is necessary and sufficient for the orthogonal coordinates to permit additive separation of the Hamilton–Jacobi equation

$$\sum_{j=1}^{n} H_j'^{-2} (\partial_j W)^2 = 0 \tag{2.10}$$

i.e. separation in the form $W = \Sigma_{i=1}^{n} W^{(i)}(x^i)$. However, conformal Stäckel form is not sufficient for product R separation of the Laplace equation. In addition we must require equality of the coefficients of ∂_j and the zeroth-order terms on each side of (2.5):

$$f_j + 2\partial_j \ln R = l_j(x^j) \tag{2.11}$$

$$R^{-1}(\Delta R) = \sum_i Q H_i^{-2} m_i(x^i). \tag{2.12}$$

Here

$$f_j \equiv \partial_j f \equiv \partial_j \ln(Q^{(2-n)/2}h/S) \qquad h = H_1 \ldots H_n. \tag{2.13}$$

Solving for R from (2.11) and substituting this expression into (2.12) we reduce the separation conditions to

$$\sum_i H_i^{-2}(f_{ii} + \tfrac{1}{2}f_i^2) = \sum_i H_i^{-2}\tilde{m}_i(x^i) \tag{2.14}$$

where $\tilde{m}_i = -2m_i + \partial_i l_i + \frac{1}{2}l_i^2$ is a function of x^i alone.

To express these conditions more simply we make use of some results from Kalnins and Miller (1978). Given a metric $\Sigma H_i^2 (\mathrm{d}x^i)^2$ in Stäckel form, we say that the function $T(\boldsymbol{x})$ is a *Stäckel multiplier* (for $\mathrm{d}s^2$) if the metric $\mathrm{d}\hat{s}^2 = T\,\mathrm{d}s^2$ is also in Stäckel form with respect to the coordinates $\{x^j\}$. It can be shown that T is a Stäckel multiplier if and only if there exist functions $k_j = k_j(x^j)$ such that

$$T(\boldsymbol{x}) = \sum_{j=1}^{n} k_j(x^j)H_j^{-2}. \tag{2.15}$$

Furthermore, necessary and sufficient conditions that T be a Stäckel multiplier are

$$\partial_{jl}T - \partial_j T \partial_l \ln H_j^{-2} - \partial_l T \partial_j \ln H_l^{-2} = 0 \qquad j \neq l. \tag{2.16}$$

Recall that necessary and sufficient conditions that $\mathrm{d}s^2$ be in Stäckel form are (Eisenhart 1949, appendix 13)

$$\partial_{ij} \ln H_l^{-2} + \partial_i \ln H_l^{-2}\,\partial_j \ln H_l^{-2} - \partial_i \ln H_l^{-2}\,\partial_j \ln H_i^{-2}$$
$$- \partial_j \ln H_l^{-2}\,\partial_i \ln H_j^{-2} = 0 \qquad i \neq j. \tag{2.17}$$

Theorem 1. Necessary and sufficient conditions that the orthogonal coordinates $\{x^i\}$ be R separable for the Laplace equation

$$\frac{1}{h'}\sum_{j=1}^{n} \partial_j(h' H_j'^{-2}\,\partial_j \psi) = 0 \qquad h' = H_1' \ldots H_n'$$

are

(i) there exists a non-zero function Q such that the metric $\mathrm{d}s^2 = Q\,\Sigma H_j'^2 (\mathrm{d}x^j)^2 = \Sigma H_j^2 (\mathrm{d}x^j)^2$ is in Stäckel form;

(ii) $\Sigma H_j^{-2}(f_{jj} + \frac{1}{2}f_j^2)$ is a Stäckel multiplier where $f_j = \partial_j \ln(Qh'/S)$ and S is the determinant of the Stäckel matrix.

If these conditions are satisfied then

$$R = \left(\frac{S}{Qh'}\right)^{1/2} \prod_{j=1}^{n} L_j(x^j) \tag{2.18}$$

where the $L_j = L_j(x^j)$ are arbitrary.

We say that the orthogonal coordinates $\{x^i\}$ are *separable* for the Laplace equation provided they are R separable with $R \equiv 1$, and *trivially R separable* if $R = \Pi_{i=1}^n L_i(x^i)$.

Now suppose that the coordinates $\{x^i\}$ R separate the Laplace equation. Then by expanding the Stäckel determinant on the ith column and using (2.4) we obtain operators A_i, $i = 1, \ldots, n$, such that $A_i\psi = -\lambda_i\psi$, $(\lambda_1 = 0)$ for an R-separated ψ:

$$A_i = \sum_j \frac{S^{ji}}{S}(L_j + \tfrac{1}{2}\partial_j[f_j - l_j] + \tfrac{1}{4}[f_j^2 - l_j^2]). \tag{2.19}$$

Here,

$$\Delta = QA_1. \tag{2.20}$$

It is convenient to introduce the functions $\rho_j^{(k)}(\boldsymbol{x})$ where

$$S^{jk}/S = \rho_j^{(k)} H_j^{-2} \qquad 1 \leqslant j, k \leqslant n. \tag{2.21}$$

Then $\rho_j^{(1)} = 1$ and as shown in Eisenhart (1949, appendix 13) or Koornwinder (1980)

$$\partial_i \rho_j^{(k)} = (\rho_i^{(k)} - \rho_j^{(k)}) \partial_i \ln H_j^{-2}. \tag{2.22}$$

Thus

$$A_i = \sum_j \rho_j^{(i)} H_j^{-2} (\partial_{jj} + f_j \partial_j + \xi_j) \qquad 1 \leqslant i \leqslant n \tag{2.23}$$

where

$$\xi_j = m_j + \tfrac{1}{2} \partial_j (f_j - l_j) + \tfrac{1}{4} (f_j^2 - l_j^2). \tag{2.24}$$

Using (2.22)–(2.24) and (2.14) one can directly verify that

$$[A_j, A_k] = 0 \qquad 1 \leqslant j, k \leqslant n \tag{2.25}$$

where $[A, B] = AB - BA$. Furthermore,

$$[A_l, \Delta] = C_l \Delta \qquad 2 \leqslant l \leqslant n \tag{2.26}$$

where $C_l = [A_l, Q]$. Thus the A_l, $l \geqslant 2$, form a commuting family of *conformal symmetry operators* for Δ with the property that their simultaneous eigenfunctions are the R-separable solutions of the Laplace equation corresponding to the coordinates $\{x^j\}$.

In Kalnins and Miller (1982a) we studied the corresponding problem for the Hamilton–Jacobi equation (2.10). We review the results of Kalnins and Miller (1982a) that are germane to our present problem. There, use was made of the natural symplectic structure on the $2n$-dimensional cotangent bundle $\tilde{V}_n$ of V_n. Corresponding to local coordinates $\{x^i\}$ on V_n there are coordinates $\{x^i, p_i\}$ on $\tilde{V}_n$. The *Poisson bracket* of functions $\mathscr{F}(\boldsymbol{x}, \boldsymbol{p})$, $\mathscr{S}(\boldsymbol{x}, \boldsymbol{p})$ on $\tilde{V}_n$ is the function

$$\{\mathscr{F}, \mathscr{S}\}(\boldsymbol{x}, \boldsymbol{p}) = \sum_l (\partial_{p_l} \mathscr{F} \partial_{x^l} \mathscr{S} - \partial_{x^l} \mathscr{F} \partial_{p_l} \mathscr{S}). \tag{2.27}$$

Let $\mathscr{H} = \Sigma_j H_j'^{-2} p_j^2$ be the Hamiltonian corresponding to (2.10). If $\{x^i\}$ is an orthogonal separable coordinate system for the Hamilton–Jacobi equation then there exists a Stäckel matrix $(S_{ji}(x^j))$ such that $H_j'^{-2}$ is given by (2.9). Also, the quadratic forms

$$\mathscr{A}_k = \sum_j \rho_j^{(k)} H_j^{-2} p_j^2 \qquad 1 \leqslant k \leqslant n \tag{2.28}$$

($\mathscr{H} = Q\mathscr{A}_1$) satisfy $\{\mathscr{A}_i, \mathscr{A}_j\} = 0$, and when evaluated for $p_j = \partial_j W$ with W a separable solution of (2.10) they satisfy $\mathscr{A}_k = \lambda_k$, ($\lambda_1 = 0$) where $\lambda_2, \ldots, \lambda_n$ are the separation parameters. Thus, the $\{\mathscr{A}_i, 2 \leqslant i \leqslant n\}$ form an involutive family of conformal Killing tensors.

Let $a^{ij}(\boldsymbol{y})$ be a symmetric contravariant 2-tensor on V_n, expressed in terms of local coordinates $\{y^k\}$, and let $g^{ij}(\boldsymbol{y})$ be the contravariant metric tensor. A *root* $\rho(\boldsymbol{y})$ of a^{ij} is an analytic solution of the characteristic equation

$$\det(a^{ij} - \rho g^{ij}) = 0 \tag{2.29}$$

and an eigenform $\omega = \Sigma_k \mu_k \, dy^k$ corresponding to ρ is a non-zero one-form such that

$$\sum_j (a^{ij} - \rho g^{ij})\mu_j = 0 \qquad 1 \leqslant i \leqslant n. \tag{2.30}$$

It follows from (2.21) that for a Hamilton–Jacobi separable system $\{x^j\}$ the $\rho_i^{(k)}$, $1 \leqslant i \leqslant n$ are the roots of the quadratic forms $\mathscr{A}_k$ with respect to the metric ds^2 and the dx^i constitute a basis of simultaneous eigenforms. In Kalnins and Miller (1982a) the authors proved the following theorem.

Theorem 2. Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system $\{x^j\}$ for the Hamilton–Jacobi equation (1.3) are that there exist $n-1$ quadratic functions $\mathscr{B}_k = \Sigma_{i,j} b^{ij}_{(k)} p_i p_j$ on $\tilde{V}_n$, $2 \leqslant k \leqslant n$, such that
(1) each $\mathscr{B}_k$ is a conformal Killing tensor;
(2) $\{\mathscr{B}_i, \mathscr{B}_j\} = 0$, $2 \leqslant i, j \leqslant n$;
(3) the set $(\mathscr{H}, \mathscr{B}_k)$ is linearly independent (as n quadratic forms);
(4) there is a basis $\{\omega_{(j)}: 1 \leqslant j \leqslant n\}$ of simultaneous eigenforms for the $\{\mathscr{B}_k\}$.
If conditions (1)–(4) are satisfied then there exist functions $f^j(\boldsymbol{x})$ such that $\omega_{(j)} = f^j \, dx^j$, $1 \leqslant j \leqslant n$.

We note that the conformal Killing tensors $\mathscr{B}_k$ cannot necessarily be identified with the Killing tensors (2.28). Consider the Euclidean space example

$$\mathscr{H} = p_1^2 + p_2^2 + p_3^2 \qquad \mathscr{B}_2 = x^1\mathscr{H} + p_1^2 \qquad \mathscr{B}_3 = p_2^2. \tag{2.31}$$

Here $\{\mathscr{B}_2, \mathscr{B}_3\} = 0$, $\mathscr{B}_2, \mathscr{B}_3$ are conformal Killing tensors and the simultaneous eigenforms are dx^1, dx^2, dx^3 (Cartesian coordinates). However, there exists no function Q such that $\{Q^{-1}\mathscr{H}, \mathscr{B}_j\} = 0$, $j = 2, 3$.

At this point it is tempting to try to mimic the proof of theorem 3 of Kalnins and Miller (1982b), a theorem which characterises R separation for the Helmholtz equation in terms of commuting symmetry operators for Δ. One would presumably show that a family of commuting conformal symmetry operators A_l, $2 \leqslant l \leqslant n$, whose associated quadratic forms determine an additively separable coordinate system $\{x^j\}$ for the Hamilton–Jacobi equation can, with the possible modification of some first derivative and zeroth-order terms, be identified with the operators (2.23), so that the coordinates $\{x^j\}$ also R separate the Laplace equation. This straightforward procedure will not work, however. The principal difficulty is this: in general, given a commuting family of conformal symmetry operators B_l, $2 \leqslant l \leqslant n$, whose associated quadratic forms determine a separable coordinate system $\{x^j\}$ for the Hamilton–Jacobi equation, it is not possible to find a function Q such that $Q^{-1}\Delta$ commutes with the B_l. Also this difficulty cannot be remedied by modifying only the first- and zeroth-order terms of the B_l. The basic cause of the problem is that $f(x)\Delta$ is a conformal symmetry of Δ for any function f and this produces a degree of ambiguity in the choice of conformal symmetries.

A modification of our previous example clarifies the issue:

$$\Delta = \partial_{11} + \partial_{22} + \partial_{33} \qquad B_2 = x^1\Delta + \partial_{11} \qquad B_3 = \partial_{22}. \tag{2.32}$$

Here $[B_2, B_3] = 0$, $[\Delta, B_2] = 2\partial_1\Delta$ and $[\Delta, B_3] = 0$ so that B_2, B_3 form a commuting family of conformal symmetries for Δ. Moreover, these operators determine a Hamilton–Jacobi and a Laplace separable coordinate system (Cartesian coordinates x^1, x^2, x^3) via the equations

$$\Delta\psi = 0 \qquad B_2\psi = \lambda_2\psi \qquad B_3\psi = \lambda_3\psi. \tag{2.33}$$

On the other hand it is not possible to find Q such that $[Q^{-1}\Delta, B_j]=0$, $j=2,3$, so that the B_j cannot be identified with the operators (2.23).

Another instructive example is provided by the Euclidean space Laplace operator again and the operators (in Cartesian coordinates)

$$B_2=J_3^2 \qquad B_3=P_3^2+K_3 \tag{2.34}$$

where

$$J_3=x^2\partial_1-x^1\partial_2 \qquad P_3=\partial_3 \qquad K_3=[(x^3)^2-(x^1)^2-(x^2)^2]\partial_3+2x^3x^1\partial_1+2x^3x^2\partial_2+x^3. \tag{2.35}$$

Here, B_2 and B_3 are commuting conformal symmetry operators and their associated quadratic forms determine a separable coordinate system (cylindrical coordinates). However, the operators $A_2=J_3^2$, $A_3=P_3^2$ are those which would be associated with cylindrical coordinates by the construction (2.19). There appears to be no natural way of eliminating the operator K_3 in advance.

3. The basic result

We can find no *a priori* grounds by which to determine the 'natural' operators $\{A_k\}$ defining an R-separable coordinate system from an arbitrary defining set $\{B_k\}$, although the $\{A_k\}$ can be obtained *a posteriori* from the coordinates. Once this ambiguity is accepted, however, a satisfying analogy to theorem 2 emerges.

Let $\{y^i\}$ be a local coordinate system on V_n and let B be a second-order conformal symmetry operator, expressed in these coordinates by

$$B=\sum_{i,j} b^{ij}\partial_{ij}+c^i\partial_i+d.$$

Then B is uniquely associated with a conformal Killing tensor $\mathcal{B}$ on $\tilde{V}_n$ and defined in local coordinates by

$$\mathcal{B}=\sum b^{ij}p_ip_j.$$

We can talk about the *roots* and *eigenforms* of B, meaning by this the roots and eigenforms of $\mathcal{B}$.

Theorem 3. Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system $\{x^i\}$ for the Laplace equation (1.1) are that there exist $n-1$ second-order differential operators $B_2, \ldots, B_n$ on V_n such that

(1) each B_k is a conformal symmetry operator;
(2) $[B_i, B_j]=0$, $2\leqslant i, j\leqslant n$;
(3) the set $(\mathcal{H}, \mathcal{B}_2, \ldots, \mathcal{B}_n)$ is linearly independent;
(4) there is a basis $\{\omega_{(j)}: 1\leqslant j\leqslant n\}$ of simultaneous eigenforms for the B_k.

If conditions (1)–(4) are satisfied then there exist functions $f^j(\boldsymbol{x})$ such that $\omega_{(j)}=f^j(\boldsymbol{x})\,\mathrm{d}x^j$, $j=1,\ldots,n$.

Proof. If $\{x^i\}$ is an R-separable coordinate system for (1.1) then, as we showed in § 2, there exists a family of operators $\{B_k\}$ satisfying conditions (1)–(4) and such that $\{\mathrm{d}x^j\}$ is a simultaneous eigenbasis for these operators.

Conversely, suppose $\{B_k\}$ is a family of operators satisfying conditions (1)–(4). Comparing coefficients of the third derivative terms in the commutators (1) and (2) we see that the associated quadratic forms $\{\mathscr{B}_k\}$ are conformal Killing tensors and that $\{\mathscr{B}_k, \mathscr{B}_l\}=0$, $2 \leqslant k, l \leqslant n$. Thus, from conditions (3) and (4), the hypotheses of theorem 2 are satisfied. It follows that there exists an orthogonal local coordinate system $\{x^j\}$ such that $\mathrm{d}x^j$ is a simultaneous eigenform for each operator B_k, and a Stäckel matrix $(S_{ji}(x^l))$ which defines an additive separation of variables for the Hamilton–Jacobi equation (1.3). Further, there exists a function Q such that in these coordinates

$$\mathscr{H}=Q \sum_{j=1}^{n} H_j^{-2} p_j^2 \qquad H_j^{-2}=S^{j1}/S. \tag{3.1}$$

Denoting the roots of B_k with respect to $B_1=Q^{-1}\Delta$ by $\rho_j^{(k)}$, $j=1,\ldots,n$ we have

$$\mathscr{B}_k=\sum_j \rho_j^{(k)} H_j^{-2} p_j^2 \qquad 1 \leqslant k \leqslant n \tag{3.2}$$

where $\rho_j^{(1)} \equiv 1$. It follows from the results of Kalnins and Miller (1982a) that

$$\partial_i(\rho_i^{(m)}-\rho_j^{(m)})=(\rho_j^{(m)}-\rho_i^{(m)})\partial_i \ln H_j^{-2} \qquad 1 \leqslant m \leqslant n \tag{3.3}$$

and that

$$\rho_i^{(k)} \partial_i \rho_i^{(l)}=\rho_i^{(l)} \partial_i \rho_i^{(k)} \qquad 1 \leqslant i \leqslant n, 2 \leqslant l, k \leqslant n. \tag{3.4}$$

By definition

$$B_1=\sum_i H_i^{-2}(\partial_{ii}+f_i \partial_i) \qquad f_i=\partial_i \ln(h' Q/S) \tag{3.5}$$

and we can write

$$B_k=\sum_i \rho_i^{(k)} H_i^{-2}(\partial_{ii}+f_i \partial_i)+\sum_i \xi_k^i \partial_i+\mu_k \qquad 2 \leqslant k \leqslant n. \tag{3.6}$$

To simplify the following computations we perform the similarity transformation

$$B \leftrightarrow \tilde{B}=\mathrm{e}^{f/2} B \mathrm{e}^{-f/2}. \tag{3.7}$$

(Note that $[B_i, B_j]^{\tilde{}}=[\tilde{B}_i, \tilde{B}_j]$.) Then,

$$\begin{gathered}
\tilde{B}_m=\sum_i \rho_i^{(m)} H_i^{-2} \partial_{ii}+\sum_i \xi_m^i \partial_i+\tilde{\mu}_m \\
\rho_i^{(1)} \equiv 1 \qquad \xi_1^i \equiv 0 \qquad \mu_1=0 \\
\tilde{\mu}_m=\mu_m-\tfrac{1}{2} \sum_i \rho_i^{(m)} H_i^{-2}(f_{ii}+\tfrac{1}{2} f_i^2)-\tfrac{1}{2} \sum_i \xi_m^i f_i .
\end{gathered} \tag{3.8}$$

Now we exploit the relations

$$[\tilde{B}_1, \tilde{B}_k]=\Big(\sum_i t_k^i \partial_i+a_k\Big) \tilde{B}_1 \qquad 2 \leqslant k \leqslant n \tag{3.9}$$

and

$$[\tilde{B}_k, \tilde{B}_l]=0 \qquad 2 \leqslant k, l \leqslant n. \tag{3.10}$$

Comparing coefficients of third-derivative terms in (3.9) we obtain (3.3) again, as well as the new relation

$$t_k^i=2 H_i^{-2} \partial_i \rho_i^{(k)}. \tag{3.11}$$

Comparing coefficients of ∂_{jj} in (3.9) we find

$$2\partial_j(\xi_j^m) - \sum_l \xi_l^m \partial_l \ln H_j^{-2} = a_m - \sum_l H_l^{-2} \partial_{ll} \rho_l^{(m)} \qquad 2 \leqslant m \leqslant n. \tag{3.12}$$

Comparison of the coefficients of ∂_l in (3.9) leads to

$$-2\partial_l \tilde{\mu}_m = \partial_l(\rho_l^{(m)} T) - H_l^2 \sum_i H_i^{-2} \partial_{ii} \xi_m^l \qquad 2 \leqslant m \leqslant n \tag{3.13}$$

where

$$T = \sum_k H_k^{-2} (f_{kk} + \tfrac{1}{2} f_k^2). \tag{3.14}$$

Comparison of the coefficients ∂_{ij}, $i \neq j$ in (3.9) yields

$$H_i^{-2} \partial_i(\xi_m^l) + H_l^{-2} \partial_l(\xi_m^i) = 0 \qquad i \neq l,\ m \geqslant 2. \tag{3.15}$$

Now define functions $\theta_l^{(m)}$ by

$$\theta_l^{(m)} = -\tfrac{1}{2} H_l^2 \sum_i H_i^{-2} \partial_{ii} \xi_m^l \qquad 1 \leqslant l \leqslant n,\ 2 \leqslant m \leqslant n. \tag{3.16}$$

We claim that $\partial_i \theta_l^{(m)} = \partial_l \theta_i^{(m)}$, i.e. that there exist functions $\alpha^{(m)}$ such that $\theta_l^{(m)} = \partial_l \alpha^{(m)}$. Indeed

$$\partial_{jl} \ln H_i^{-2} = \partial_{lj} \ln H_j^{-2} \tag{3.17}$$

since H_i^{-2} is in Stäckel form (see (2.17)), and using (3.12), (3.15), (3.17) we can write (3.16) in the form

$$-4\theta_l^{(m)} = \partial_l \left(a_m - \sum_i (H_i^{-2} \partial_{ii} \rho_i^{(m)} + \xi_m^i \partial_i \ln H_l^{-2}) \right) + \sum_{k \neq l} (\partial_l \xi_m^k \partial_k \ln H_k^{-2} - \partial_l \xi_m^k \partial_k \ln H_l^{-2}) - \partial_{ll} \xi_m^l. \tag{3.18}$$

Denoting the left-hand side of (3.12) by $\psi_j^{(m)}$, we have $\partial_j \psi_l^{(m)} = \partial_l \psi_j^{(m)}$, $l \neq j$, and this identity, together with (3.17), yields

$$\Theta_{jl}^{(m)} = \Theta_{lj}^{(m)} \qquad l \neq j$$

$$\Theta_{jl}^{(m)} = 2\partial_{jll} \xi_m^l - \sum_i (\partial_{jl} \xi_m^i \partial_i \ln H_l^{-2} + \partial_j \xi_m^i \partial_{li} \ln H_l^{-2} + \partial_l \xi_m^i \partial_{ji} \ln H_l^{-2}). \tag{3.19}$$

It follows directly from (3.18), (3.19) that $\partial_j \theta_l^{(m)} = \partial_l \theta_j^{(m)}$. Thus $\theta_l^{(m)} = \partial_l \alpha^{(m)}$. Substituting this result in (3.13) we see that there must exist a function $c^{(m)}$ such that

$$\partial_l c^{(m)} = \partial_l (\rho_l^{(m)} T). \tag{3.20}$$

Then $\partial_{jl} c^{(m)} = \partial_{lj} c^{(m)}$, or

$$\partial_{jl} T = \partial_j T \partial_l \ln H_j^{-2} + \partial_l T \partial_j \ln H_l^{-2} \qquad j \neq l \tag{3.21}$$

where we have used condition (3), (3.3), and relations (2.17). Comparing this result with (2.16), we see that

$$T = \sum_k H_k^{-2} (f_{kk} + \tfrac{1}{2} f_k^2)$$

is a Stäckel multiplier. Hence, by theorem 1, the coordinates $\{x^i\}$ are R separable for the Laplace equation.

We have not used all of the information contained in the commutation relations for the $\{B_k\}$. One can additionally show that the first-order terms $\Sigma_l \xi^l_k \partial_l$ are extraneous and can be deleted. Indeed, define operators C_k by

$$\tilde{B}_k = \tilde{C}_k + \sum_l \xi^l_k \partial_l + \alpha^{(k)} \qquad 1 \leq k \leq n \tag{3.22}$$

where $\alpha^{(1)} = 0$. Then, employing relations (3.3) and (3.4), we can directly verify that $[\tilde{C}_k, \tilde{C}_m] = 0$ for $2 \leq k, m \leq n$. Furthermore, comparison of the zeroth-order terms in (3.9) yields

$$\sum_i H_i^{-2} (\partial_{ii}\tilde{\mu}_m + \tfrac{1}{2}\rho_i^{(m)} \partial_{ii} T + \partial_i \rho_i^{(m)} \partial_i T) + \tfrac{1}{2} \sum_i \xi^i_m \partial_i T + \tfrac{1}{2} a_m T = 0 \qquad 2 \leq m \leq n \tag{3.23}$$

and this is sufficient to show that the $\{C_k\}$ are conformal symmetry operators. Thus the 'reduced' operators C_k can be used to define the R separation. This argument suffices to eliminate the operator K_3 in example (2.34).

The operators C_k can be used to characterise the separated solutions. Indeed, from lemma 1 of Kalnins and Miller (1982a) we can assume, without loss of generality, that $H_n^{-2} = 1$. Then setting $\rho_i^{(m)} = \beta_i^{(m)} - \rho_n^{(m)}$, $1 \leq m \leq n$, and substituting this expression in equations (3.3) we find

$$\partial_i \beta_j^{(m)} = (\beta_i^{(m)} - \beta_j^{(m)}) \partial_i \ln H_j^{-2} \qquad 1 \leq i, j, m \leq n. \tag{3.24}$$

Note that equations (3.24) are identical with (2.22). Setting

$$A_m = C_m - \rho_n^{(m)} C_1 \qquad 1 \leq m \leq n \tag{3.25}$$

where $\rho_n^{(1)} = 0$ one can easily verify that the operators A_m can be identified with (2.23). Here $A_m\psi = C_m\psi$ for any solution ψ of $\Delta\psi = 0$ since $C_1 = Q^{-1}\Delta$. Thus the separated solutions corresponding to the coordinates $\{x^k\}$ can be characterised by

$$C_m \psi = -\lambda_m \psi \qquad \lambda_1 = 0. \tag{3.26}$$

We can also use the proof of the previous theorem to generalise theorem 3 in Kalnins and Miller (1982b).

Corollary. Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system $\{x^j\}$ for the Helmholtz equation $\Delta\psi = E\psi$, $E \neq 0$, are that there exist n second-order differential operators $B_1 = \Delta, B_2, \ldots, B_n$ such that

(1) $[B_i, B_j] = 0$, $1 \leq i, j \leq n$;

(2) the set $\{\mathcal{B}_i\}$ is linearly independent;

(3) there is a basis $\{\omega_{(j)}: 1 \leq j \leq n\}$ of simultaneous eigenforms for the B_k.

If these conditions are satisfied then there exist functions $f^j(\boldsymbol{x})$ such that $\omega_{(j)} = f^j \, \mathrm{d}x^j$, $1 \leq j \leq n$.

Proof. This follows from the proof of theorem 3 with $Q = 1$, $\partial_i \rho_i^{(k)} = 0$, $t^i_k = a_k = 0$.

This generalises the principal result of Kalnins and Miller (1982b) in the sense that we do not have to require that the operators B_k are in self-adjoint form. However, we can no longer assert that the R-separated solutions of the Helmholtz equation are the simultaneous eigenfunctions of the operators B_k. An example is provided by the

Laplace operator (2.32) and

$$B_2 = J_1^2 + J_2^2 + J_3^2 + P_3 \qquad B_3 = J_3^2$$

where J_3 and P_3 are given by (2.35) (spherical coordinates).

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